

VARIATION OF LOEWNER CHAINS, EXTREME AND SUPPORT POINTS IN THE CLASS S^0 IN HIGHER DIMENSIONS

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ABSTRACT. We introduce a family of natural normalized Loewner chains in the unit ball, which we call “geräumig”—spacious—which allow to construct, by means of suitable variations, other normalized Loewner chains which coincide with the given ones from a certain time on. We apply our construction to the study of support points, extreme points and time-log M -reachable functions in the class S^0 of mappings admitting parametric representation.

1. INTRODUCTION

Let $\mathbb{B}^n := \{z \in \mathbb{C}^n : \|z\|^2 < 1\}$ denote the Euclidean unit ball of \mathbb{C}^n . Let

$$S := \{f : \mathbb{B}^n \rightarrow \mathbb{C}^n : f(0) = 0, df_0 = \text{id}, f \text{ univalent}\}$$

be the class of normalized univalent mappings in \mathbb{B}^n . For $n = 1$ the class S is compact, and every $f \in S$ can be embedded into a normalized Loewner chain (see [27]). A. C. Schaeffer (see, e.g. [34]) proved that support points—namely those $f \in S$ which maximize a non-constant bounded linear operator—in dimension one are slit functions. A similar result also holds for extreme points of the class S (see e.g. [8]).

In higher dimension, the class S is not compact, and it is not known whether every element in S can be embedded into a normalized Loewner chain. Partial results on this can be found in [4], [11], [18].

For $n > 1$ the compact subclass S^0 of S of mappings admitting parametric representation was introduced in [11]; it was first considered by Poreda (see [28], [29]) on the polydisc. Support points and extreme points for such a class have been studied since then (see [15], [16], [20], [35]). In the case of one complex variable, see [32] and the references therein.

One of the main difficulties when dealing with univalent mappings in higher dimension is that the lack of an uniformization theorem does not allow to construct easily variations of a given normalized Loewner chain.

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The aim of the present paper is to define a natural class of normalized Loewner chains, which we call *geräumig*, which allow to construct other normalized Loewner chains having the property that from a certain time on, they coincide with the initial *geräumig* Loewner chain.

This variational method seems to be completely new and seems to adapt well to the case of bounded univalent mappings of the ball having some regular extension up to the boundary.

We refer the reader to Section 2 for the definition of “*geräumig*” Loewner chains and Theorem 3.1 for the result on variation of “*geräumig*” Loewner chains. For the time being, we content ourselves to give the following definition. A normalized Loewner chain $(f_t)_{t \geq 0}$ on \mathbb{B}^n is called *exponentially squeezing* in $[T_1, T_2)$ for some $0 \leq T_1 < T_2 \leq \infty$ provided there exists $a \in (0, 1)$ such that for all $T_1 \leq s < t < T_2$ it follows $\|f_t^{-1}(f_s(z))\| \leq e^{a(s-t)}\|z\|$, for all $z \in \mathbb{B}^n$.

In particular we have the following result (whose proof is in Section 4):

Proposition 1.1. *Let $(f_t)_{t \geq 0}$ be a normal Loewner chain which is exponentially squeezing in $[T_1, T_2)$ for some $0 \leq T_1 < T_2 \leq +\infty$. Then f_0 is not a support point of S^0 . Also, f_0 is not an extreme point of S^0 .*

In fact, an exponentially squeezing normal Loewner chain can be suitably re-parameterized in time in order to construct a *geräumig* Loewner chain (see Theorem 2.23). It is interesting to note that all bounded normalized functions in the unit disc can be embedded into an exponentially squeezing chain (and in fact *geräumig* Loewner chain) from a certain time on—which, geometrically, amounts to evolve the image of the mapping into a disc in a finite time and consider then the natural radial dilatation of such a disc. While, in higher dimension, this is no longer the case (see Example 3.2).

Proposition 1.1 allows to prove directly the following results (precise definitions and the proofs are contained in Section 4):

Theorem 1.2. *Let $f \in S^0$. Assume that*

- (1) *either f is an almost starlike mapping of order > 0 ,*
- (2) *or $\sup_{z \in \mathbb{B}^n} \|df_z - \text{id}\| < 1$,*
- (3) *or there exists $g \in S^0$ and $r \in (0, 1)$ such that $f(z) = \frac{1}{r}g(rz)$, for all $z \in \mathbb{B}^n$,*

Then f is not a support point of the class S^0 . Also, f is not an extreme point of S^0 .

Finally, in Section 5 we apply our results to study time-log M -reachable mappings and their geometric counterparts, the mappings that can be evolved in finite time to a ball. As a result, and in neat contrast to the one-dimensional case, we find an example of a family of mappings $\Phi^N \in S^0$, $N > 2$, which are bounded by a constant $M > 1$, are not support points, nor extreme points of S^0 but cannot be reached in time $\log R$ for all $2 < R < N$. Those mappings Φ^N are however reachable in time $\log N$ and are in fact support points of the set of time $\log N$ -reachable mappings (see Theorem 5.8).

2. GERÄUMIG LOEWNER CHAINS

2.1. Regular families, subordination chains and Loewner chains. In what follows we denote by \mathbb{R}^+ the semigroup of nonnegative real numbers and by \mathbb{N} the semigroup of nonnegative integer numbers.

Let

$$\mathcal{M} := \{h \in \text{Hol}(\mathbb{B}^n, \mathbb{C}^n) : h(0) = 0, dh_0 = \text{id}, \text{Re} \langle h(z), z \rangle > 0, \forall z \in \mathbb{B}^n \setminus \{0\}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in \mathbb{C}^n . Applications of this family in the study of biholomorphic mappings on \mathbb{B}^n and the Loewner theory in higher dimensions may be found in [1], [4], [6], [9], [10], [11], [17, Chapter 8], [25], [36], [37].

Remark 2.1. If $h \in \text{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ is such that $h(0) = 0$, $dh_0 = \text{id}$, $\text{Re} \langle h(z), z \rangle \geq 0$ for all $z \in \mathbb{B}^n$, then, in fact, $h \in \mathcal{M}$. Indeed, assume that for some $z_0 \in \mathbb{B}^n \setminus \{0\}$ it holds $\text{Re} \langle h(z_0), z_0 \rangle = 0$. Let $v := z_0 / \|z_0\|$ and set $g(\zeta) := \langle h(\zeta v), v \rangle$ for $\zeta \in \mathbb{D}$, where \mathbb{D} is the unit disc in \mathbb{C} . Then it is easy to see that $g(\zeta) = \zeta p(\zeta)$ with $p : \mathbb{D} \rightarrow \mathbb{C}$ holomorphic, $p(0) = 1$ and $\text{Re} p(\zeta) \geq 0$. But $\text{Re} p(\|z_0\|) = 0$, which would force $\text{Re} p(\zeta) \equiv 0$, contradiction. Hence $\text{Re} \langle h(z), z \rangle > 0$ for all $z \in \mathbb{B}^n \setminus \{0\}$ and thus $h \in \mathcal{M}$.

Definition 2.2. A *Herglotz vector field* associated with the class \mathcal{M} on \mathbb{B}^n is a mapping $G : \mathbb{B}^n \times \mathbb{R}^+ \rightarrow \mathbb{C}^n$ with the following properties:

- (i) The mapping $G(z, \cdot)$ is measurable on \mathbb{R}^+ for all $z \in \mathbb{B}^n$.
- (ii) $-G(\cdot, t) \in \mathcal{M}$ for a.e. $t \in [0, +\infty)$.

Remark 2.3. Due to the estimates for the class \mathcal{M} (see [17, Theorem 7.1.7]), a Herglotz vector field associated with the class \mathcal{M} on \mathbb{B}^n is an L^∞ -Herglotz vector field on \mathbb{B}^n in the sense of [6].

Definition 2.4. A family $(f_t)_{t \geq 0}$ of holomorphic mappings from \mathbb{B}^n to \mathbb{C}^n such that $f_t(0) = 0$ and $d(f_t)_0 = e^t \text{id}$ for all $t \geq 0$, is called a *normalized regular family* if

- (i) the mapping $t \mapsto f_t$ is continuous with respect to the topology in $\text{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ induced by the uniform convergence on compacta in \mathbb{B}^n ,
- (ii) there exists a set of zero measure $N \subset [0, +\infty)$ such that for all $t \in [0, +\infty) \setminus N$ and all $z \in \mathbb{B}^n$ the partial derivative $\frac{\partial f_t}{\partial t}(z)$ exists and it is holomorphic.

For a given Herglotz vector field $G(z, t)$ associated with the class \mathcal{M} on \mathbb{B}^n , a *normalized solution* to the *Loewner-Kufarev PDE* associated to $G(z, t)$ consists of a normalized regular family $(f_t)_{t \geq 0}$ such that the following equation is satisfied for a.e. $t \geq 0$ and for all $z \in \mathbb{B}^n$

$$(2.1) \quad \frac{\partial f_t}{\partial t}(z) = -d(f_t)_z \cdot G(z, t).$$

Definition 2.5. A *normalized subordination chain* $(f_t)_{t \geq 0}$ is a family of holomorphic mappings $f_t : \mathbb{B}^n \rightarrow \mathbb{C}^n$, such that $f_t(0) = 0$, $d(f_t)_0 = e^t \text{id}$ for all $t \geq 0$, and for every

$0 \leq s \leq t$ there exists $\varphi_{s,t} : \mathbb{B}^n \rightarrow \mathbb{B}^n$ holomorphic such that $f_s = f_t \circ \varphi_{s,t}$. A normalized subordination chain $(f_t)_{t \geq 0}$ is called a *normalized Loewner chain* if for all $t \geq 0$ the mapping f_t is univalent.

Definition 2.6. A normalized Loewner chain $(f_t)_{t \geq 0}$ on \mathbb{B}^n is called a *normal Loewner chain* if the family $\{e^{-t}f_t(\cdot)\}_{t \geq 0}$ is normal.

Putting together [17, Chapter 8], [19], [3, Prop. 2.6] (see also [2], [9]), we have the following result:

Theorem 2.7. (1) If $(f_t)_{t \geq 0}$ is any normalized Loewner chain on \mathbb{B}^n , then it is a normalized solution to a Loewner-Kufarev PDE (2.1) for some Herglotz vector field $G(z, t)$ associated with the class \mathcal{M} in \mathbb{B}^n .
 (2) Let $G(z, t)$ be a Herglotz vector field associated with the class \mathcal{M} on \mathbb{B}^n . Then there exists a unique normal Loewner chain $(g_t)_{t \geq 0}$ —called the canonical solution—which is a normalized solution to (2.1). Moreover, $\bigcup_{t \geq 0} g_t(\mathbb{B}^n) = \mathbb{C}^n$.
 (3) If $(f_t)_{t \geq 0}$ is a normalized solution to (2.1), then $(f_t)_{t \geq 0}$ is a normalized subordination chain on \mathbb{B}^n . Moreover, there exists a holomorphic mapping $\Phi : \mathbb{C}^n \rightarrow \bigcup_{t \geq 0} f_t(\mathbb{B}^n)$, with $\Phi(0) = 0$ and $d\Phi_0 = \text{id}$ such that $f_t = \Phi \circ g_t$, where $(g_t)_{t \geq 0}$ is the canonical solution to (2.1). In particular, $(f_t)_{t \geq 0}$ is a normalized Loewner chain if and only if Φ is univalent.

We close this section with the notion of parametric representation on \mathbb{B}^n (see [11]; cf. [28], [29], on the unit polydisc in \mathbb{C}^n).

Definition 2.8. Let $f \in S$. We say that f admits *parametric representation* if

$$f(z) = \lim_{t \rightarrow \infty} e^t \varphi(z, t)$$

locally uniformly on \mathbb{B}^n , where $\varphi(z, 0) = z$ and

$$(2.2) \quad \frac{\partial \varphi}{\partial t}(z, t) = G(\varphi(z, t), t), \quad \text{a.e. } t \geq 0, \quad \forall z \in \mathbb{B}^n,$$

for some Herglotz vector field G associated with the class \mathcal{M} on \mathbb{B}^n .

We denote by S^0 the subset of S consisting of mappings which admit parametric representation.

Remark 2.9. (i) It was shown in [18] (see also [11]; cf. [28], [29]) that $f \in S$ has parametric representation if and only if there exists a normal Loewner chain $(f_t)_{t \geq 0}$ on \mathbb{B}^n such that $f_0 = f$.

(ii) It is known that S^0 is compact in the topology of uniform convergence on compacta and that $S^0 \neq S$ (see [18]; see also [11] and [17]).

2.2. Exponentially squeezing and geräumig Loewner chains. We start with a proposition:

Proposition 2.10. *Let $(f_t)_{t \geq 0}$ be a normalized Loewner chain on \mathbb{B}^n . Let $0 \leq T_1 < T_2 \leq \infty$ and $a \in (0, 1)$. The following conditions are equivalent:*

(1) *For a.e. $t \in [T_1, T_2)$ and for all $z \in \mathbb{B}^n \setminus \{0\}$ it holds*

$$(2.3) \quad \operatorname{Re} \left\langle [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z), \frac{z}{\|z\|^2} \right\rangle \geq a.$$

(2) *For all $T_1 \leq s < t < T_2$ it holds*

$$(2.4) \quad \|f_t^{-1}(f_s(z))\| \leq e^{a(s-t)} \|z\|, \quad \text{for all } z \in \mathbb{B}^n.$$

Moreover, if one of the previous conditions—and hence both—is satisfied then f_t is bounded for all $t \in [0, T_2)$ and $f_s(\mathbb{B}^n) \subset f_t(\mathbb{B}^n)$ for all $T_1 \leq s < t < T_2$.

Proof. Let $(\varphi_{s,t} := f_t^{-1} \circ f_s)_{0 \leq s \leq t}$ be the evolution family associated to $(f_t)_{t \geq 0}$ (see, e.g., [17], [6]) and let $G(z, t) = -[d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z)$ be the associated Herglotz vector field. Then $(\varphi_{s,t})$ is the unique solution to the Loewner ODE

$$(2.5) \quad \frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z), t), \quad \text{a.e. } t \geq s, \quad \forall z \in \mathbb{B}^n,$$

such that $\varphi_{s,s}(z) = z$. Assume first that (2.4) holds. Let $T_1 \leq s < t < T_2$. Fix $\eta > 0$ and let $w = \varphi_{s,t}(z)$. We have

$$(2.6) \quad \frac{\varphi_{s,t+\eta}(z) - \varphi_{s,t}(z)}{\eta} = \frac{\varphi_{t,t+\eta}(w) - w}{\eta}, \quad z \in \mathbb{B}^n, \quad t \in (s, T_2).$$

Since the limit on the left-hand side of (2.6) exists for $\eta \rightarrow 0^+$ and is equal to $\frac{\partial \varphi_{s,t}}{\partial t}(z)$ for a.e. $t \geq s$, the limit of the right-hand side of (2.6) also exists for $\eta \rightarrow 0^+$. Using (2.5) and (2.6), we conclude that

$$\lim_{\eta \rightarrow 0^+} \frac{\varphi_{t,t+\eta}(w) - w}{\eta} = G(\varphi_{s,t}(z), t), \quad \forall z \in \mathbb{B}^n, \quad \text{a.e. } t \in (s, T_2).$$

On the other hand, since $\|\varphi_{t,t+\eta}(w)\| \leq e^{-a\eta} \|w\|$, in view of the above relation,

$$(2.7) \quad \operatorname{Re} \langle G(\varphi_{s,t}(z), t), \varphi_{s,t}(z) \rangle \leq -a \|\varphi_{s,t}(z)\|^2, \quad \forall z \in \mathbb{B}^n, \quad \text{a.e. } t \in (s, T_2).$$

Let \mathbb{Q}_+ be the set of nonnegative rational numbers and let λ be the usual Lebesgue measure in \mathbb{R} . Then for each $s_k \in \mathbb{Q}_+ \cap (T_1, T_2)$, there exists $N_k \subset (s_k, T_2)$ such that $\lambda(N_k) = 0$ and

$$(2.8) \quad \operatorname{Re} \langle G(\varphi_{s_k,t}(z), t), \varphi_{s_k,t}(z) \rangle \leq -a \|\varphi_{s_k,t}(z)\|^2, \quad \forall t \in (s_k, T_2) \setminus N_k,$$

by (2.7). Let $N = \bigcup_{k \in \mathbb{N}} N_k$. Then $\lambda(N) = 0$ and if $t \in (T_1, T_2) \setminus N$ is fixed, we deduce in view of (2.8) that

$$\operatorname{Re} \langle G(\varphi_{s_k, t}(z), t), \varphi_{s_k, t}(z) \rangle \leq -a \|\varphi_{s_k, t}(z)\|^2, \quad z \in \mathbb{B}^n, s_k \in \mathbb{Q}_+ \cap (T_1, T_2), s_k < t, k \in \mathbb{N}.$$

Further, letting $\{s_{\nu(k)}\}_{k \in \mathbb{N}} \subset \mathbb{Q}_+ \cap (T_1, T_2)$, be an increasing sequence which converges to t in the above relation and using the fact that $s \mapsto \varphi_{s, t}(z)$ is continuous on $[0, t]$, we conclude that $\operatorname{Re} \langle G(z, t), z \rangle \leq -a \|z\|^2$, $z \in \mathbb{B}^n$ for all $t \in (T_1, T_2) \setminus N$. Thus, (2.3) holds.

Conversely, assume that $(f_t)_{t \geq 0}$ satisfies (2.3). Fix $z \in \mathbb{B}^n \setminus \{0\}$. Let $t_0 \in (T_1, T_2)$. Let $s \in [T_1, t_0]$. Note that for a.e. $t \geq s$ we have

$$(2.9) \quad \frac{\partial \|\varphi_{s, t}(z)\|^2}{\partial t} = 2 \operatorname{Re} \left\langle \frac{\partial \varphi_{s, t}}{\partial t}(z), \varphi_{s, t}(z) \right\rangle = 2 \operatorname{Re} \langle G(\varphi_{s, t}(z), t), \varphi_{s, t}(z) \rangle.$$

Then for a.e. $t \in [s, t_0]$ by (2.3) and (2.9), we have $\frac{\partial \|\varphi_{s, t}(z)\|^2}{\partial t} / \|\varphi_{s, t}(z)\|^2 \leq -2a$. Integrating in t between s and t_0 , we obtain $\|\varphi_{s, t_0}(z)\|^2 \leq e^{2a(s-t_0)} \|z\|^2$. Therefore, $\|\varphi_{s, t_0}(z)\| \leq e^{a(s-t_0)} \|z\|$ for all $s \in [T_1, t_0]$. This implies (2.4).

Finally note that for $T_1 \leq s < t < T_2$,

$$\overline{f_s(\mathbb{B}^n)} = \overline{f_t(\varphi_{s, t}(\mathbb{B}^n))} \subset \overline{f_t(e^{a(s-t)} \mathbb{B}^n)} = \overline{f_t(e^{a(s-t)} \mathbb{B}^n)}.$$

Hence $\overline{f_s(\mathbb{B}^n)} \subset f_t(\mathbb{B}^n)$ for all $T_1 \leq s < t < T_2$. Moreover, since $f_t(\mathbb{B}^n) \subseteq f_{T_1}(\mathbb{B}^n)$ for all $t \in [0, T_1]$, it follows also that $f_t(\mathbb{B}^n)$ is bounded for all $t \in [0, T_2]$. \square

Definition 2.11. Let $(f_t)_{t \geq 0}$ be a normalized Loewner chain in \mathbb{B}^n . We say that $(f_t)_{t \geq 0}$ is *exponentially squeezing in $[T_1, T_2]$* , for $0 \leq T_1 < T_2 \leq +\infty$ (with squeezing ratio $a \in (0, 1)$) if condition (2.3)—or equivalently (2.4)—holds.

Remark 2.12. We note that the condition (2.3) plays an important role in quasiconformal extensions of the first element f_0 of a normal Loewner chain $(f_t)_{t \geq 0}$ such that f_t is quasiregular in \mathbb{B}^n for all $t \geq 0$ (see [7], [22], [23], [24], [26]).

Remark 2.13. The condition (2.3) is related to that of a g -Loewner chain [11]. Indeed, let $(f_t)_{t \geq 0}$ be a normal Loewner chain. Then $(f_t)_{t \geq 0}$ is exponentially squeezing in $[0, \infty)$ with squeezing ratio $a \in (0, 1)$ if and only if $(f_t)_{t \geq 0}$ is a g -Loewner chain, where $g(\zeta) = \frac{1+(1-2a)\zeta}{1-\zeta}$, $|\zeta| < 1$, in view of [11] and [18]. Applications of g -Loewner chains to univalence on \mathbb{B}^n , where g is a univalent function on the unit disc which satisfies certain assumptions, may be found in [11], [13] and [17].

Example 2.14. Given $0 \leq T_1 < T_2 \leq +\infty$, examples of normal Loewner chains which are exponentially squeezing in $[T_1, T_2]$ can be constructed as follows. Let $G_1(z) = -z$ and let $G_2(z) = -(z_1 p_1(z_1), \dots, z_n p_n(z_n))$ where $p_j : \mathbb{D} \rightarrow \mathbb{C}$ are holomorphic functions such that $\operatorname{Re} p_j > 0$ for $j = 1, \dots, n$. Let $\theta : \mathbb{R}^+ \rightarrow [0, 1]$ be any measurable function such that $\theta(t) \equiv 1$ for all $t \in [T_1, T_2]$. For instance one can take $\theta(t) = 0$ for $t \in \mathbb{R}^+ \setminus [T_1, T_2]$, or, if $T_1 > 0$, one can take θ to be a C^∞ function with compact support in $(T_1 - \epsilon, T_2 + \epsilon)$

for $\epsilon > 0$ very small. Then define $G(z, t) := \theta(t)G_1(z) + (1 - \theta(t))G_2(z)$. It is easy to see that G is a Herglotz vector field associated with the class \mathcal{M} . Then by construction, the canonical solution (g_t) to (2.1) is a normal Loewner chain which is exponentially squeezing in $[T_1, T_2]$. Notice also that (g_t) might not be exponentially squeezing in $\mathbb{R}^+ \setminus [T_1, T_2]$, as is the case if p_1 is the Cayley transform and $\theta(t) = 0$ for $t \in \mathbb{R}^+ \setminus [T_1, T_2]$.

In order to properly introduce geräumig Loewner chains, we need some preliminaries of linear algebra.

Definition 2.15. Let A be an $n \times n$ matrix. We let

$$\mu(A) := \min_{\|v\|=1} \|A(v)\|.$$

Lemma 2.16. Let A be an invertible $n \times n$ matrix. Then

$$\mu(A) = \frac{1}{\|A^{-1}\|}.$$

Proof. Let $v \in \mathbb{C}^n$ be such that $\|v\| = 1$ and $A^{-1}v = \|A^{-1}\|w$ for some $w \in \mathbb{C}^n$ with $\|w\| = 1$. Hence, $\|Aw\| = \|A^{-1}\|^{-1} \geq \mu(A)$. Conversely, let $v \in \mathbb{C}^n$ be such that $\|v\| = 1$ and $Av = \mu(A)w$ for some $w \in \mathbb{C}^n$ with $\|w\| = 1$. Hence $\|A^{-1}\| \geq \|A^{-1}w\| = \mu(A)^{-1}$. \square

Remark 2.17. It is clear that $\mu(A)^n \leq |\det A|$, while a reverse inequality is not possible, as the diagonal 2×2 -matrix with entries $1/k, k$ shows. However, a matrix A is invertible if and only if $\mu(A) > 0$.

Definition 2.18. Let $(f_t)_{t \geq 0}$ be a normalized Loewner chain on \mathbb{B}^n . We say that $(f_t)_{t \geq 0}$ is *geräumig*¹ in $[T_1, T_2]$, for some $0 \leq T_1 < T_2 \leq +\infty$, if there exist $a, b > 0$ such that

- (1) for all $t \in [T_1, T_2]$ and for all $z \in \mathbb{B}^n$ it holds $\mu(d(f_t)_z) \geq a$,
- (2) for a.e. $t \in [T_1, T_2]$ and for all $z \in \mathbb{B}^n$ it holds $\left\| \frac{\partial f_t}{\partial t}(z) \right\| \leq b$,
- (3) $(f_t)_{t \geq 0}$ is exponentially squeezing in $[T_1, T_2]$.

We say that $(f_t)_{t \geq 0}$ is *geräumig* if it is *geräumig* in $[0, +\infty)$.

Remark 2.19. Let a, b be as in Definition 2.18 and assume that the squeezing ratio of $(f_t)_{t \geq 0}$ is $c \in (0, 1)$. Then

$$c \leq \operatorname{Re} \left\langle [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z), \frac{z}{\|z\|^2} \right\rangle \leq \|[d(f_t)_z]^{-1}\| \left\| \frac{\partial f_t}{\partial t}(z) \right\| \frac{1}{\|z\|} \leq \frac{b}{a}.$$

Remark 2.20. Let $(f_t)_{t \geq 0}$ be a normalized Loewner chain which satisfies the Loewner-Kufarev PDE (2.1). If $(f_t)_{t \geq 0}$ satisfies conditions (1) and (3) of Definition 2.18 and moreover there exists $c > 0$ such that

- (2') for a.e. $t \in [T_1, T_2]$ and for all $z \in \mathbb{B}^n$ it holds $\|d(f_t)_z\| \leq c$ and $\|G(z, t)\| \leq c$,

¹“geräumig” is a German word which means “spacious”

then by the Loewner-Kufarev PDE it is easy to see that $(f_t)_{t \geq 0}$ satisfies also (2) of Definition 2.18 and it is therefore geräumig in $[T_1, T_2)$.

Remark 2.21. Suppose $(f_t)_{t \geq 0}$ is a normalized Loewner chain, *respectively* a normal Loewner chain, on \mathbb{B}^n , which is geräumig in $[T_1, T_2)$ for some $0 < T_1 < T_2 \leq +\infty$. Let $\tilde{f}_t(z) := e^{-T_1} f_{T_1+t}(z)$ for $z \in \mathbb{B}^n$. Then it is easy to check that $(\tilde{f}_t)_{t \geq 0}$ is a normalized Loewner chain, *respectively* a normal Loewner chain, on \mathbb{B}^n , which is geräumig in $[0, T_2 - T_1)$ (where, if $T_2 = +\infty$, we set $T_2 - T_1 = +\infty$) and $\tilde{f}_0 = e^{-T_1} f_{T_1}$.

Example 2.22. Let $0 < T_1 < T_2 < +\infty$ and let $0 < \epsilon < T_2 - T_1$. We construct a normal Loewner chain on \mathbb{B}^2 which is geräumig in $[T_1, T_2 - \epsilon)$ but it is not geräumig in $\mathbb{R}^+ \setminus [T_1, T_2)$. Let $\theta : \mathbb{R}^+ \rightarrow [0, 1]$ be such that $\theta(t) = 1$ for $t \in [T_1, T_2]$ and $\theta(t) = 0$ in $\mathbb{R}^+ \setminus [T_1, T_2]$. Define $G(z, t) = (-\theta(t)z_1 - (1 - \theta(t))(z_1 - z_1^2), -z_2)$. Then $G(z, t)$ is a Herglotz vector field associated with the class \mathcal{M} . From the Loewner ODE $\frac{\partial \varphi_{s,t}}{\partial t}(z) = G(\varphi_{s,t}(z), t)$, we find that for $s \in [T_1, T_2]$ it holds $\varphi_{s,T_2}(z) = e^{s-T_2}z$, while, for $t > T_2$

$$\varphi_{T_2,t}(z) = e^{T_2-t} \left(\frac{z_1}{1 + (e^{T_2-t} - 1)z_1}, z_2 \right).$$

Therefore for $s \in [T_1, T_2]$ and $t > T_2$

$$\varphi_{s,t}(z) = \varphi_{T_2,t} \circ \varphi_{s,T_2}(z) = e^{s-t} \left(\frac{z_1}{1 + (e^{s-t} - e^{s-T_2})z_1}, z_2 \right).$$

By [17, Thm. 8.1.5], the canonical solution to the Loewner PDE associated with $G(z, t)$ is given by $(f_s)_{s \geq 0}$ with $f_s = \lim_{t \rightarrow \infty} e^t \varphi_{s,t}$. Hence, for $s \in [T_1, T_2]$,

$$f_s(z) = \lim_{t \rightarrow \infty} e^t \varphi_{s,t}(z) = e^s \left(\frac{z_1}{1 - e^{s-T_2}z_1}, z_2 \right).$$

From this it follows easily that there exist $a, c > 0$ such that $a \leq \mu(d(f_s)_z)$ and $\|(df_s)_z\| \leq c$ for all $z \in \mathbb{B}^2$ and for all $s \in [T_1, T_2 - \epsilon]$. Since $\|G(z, t)\| \leq 2$ for all $t \in \mathbb{R}^+$ and $z \in \mathbb{B}^2$, by the Loewner PDE and Remark 2.20, it follows that $(f_s)_{s \geq 0}$ is geräumig in $[T_1, T_2 - \epsilon)$. However, since $\lim_{z \rightarrow (1,0)} \operatorname{Re} \langle G(z, t), z \rangle = 0$ for all $t \in \mathbb{R}^+ \setminus [T_1, T_2]$, the normal Loewner chain $(f_s)_{s \geq 0}$ is not exponentially squeezing in $\mathbb{R}^+ \setminus [T_1, T_2)$, hence it is not geräumig in $\mathbb{R}^+ \setminus [T_1, T_2)$.

Theorem 2.23. Assume that $(f_t)_{t \geq 0}$ is a normalized Loewner chain, *respectively* a normal Loewner chain, on \mathbb{B}^n . If $(f_t)_{t \geq 0}$ is exponentially squeezing in $[T_1, T_2)$ for some $0 \leq T_1 < T_2 < \infty$, then there exists a normalized Loewner chain, *respectively* a normal Loewner chain, $(g_t)_{t \geq 0}$ on \mathbb{B}^n with $g_t = f_t$ for $t \in [0, +\infty) \setminus (T_1, T_2)$ (in particular, $g_0 = f_0$) and such that it is geräumig in $[T'_1, T'_2)$ for every $T_1 < T'_1 < T'_2 < T_2$.

Proof. Let $a \in (0, 1)$ be the squeezing ratio of $(f_t)_{t \geq 0}$ in $[T_1, T_2)$. Let $A \in (0, a)$. Let $\alpha : \mathbb{R}^+ \rightarrow [-(T_2 - T_1)/2, 0]$ be an absolutely continuous function such that $\alpha(t) = 0$ for

$t \in [0, T_1] \cup [T_2, +\infty)$, $\alpha(t) = -A(t - T_1)$ for $t \in (T_1, T_1 + (T_2 - T_1)/2)$ and $\alpha(t) = A(t - T_2)$ for $t \in [T_1 + (T_2 - T_1)/2, T_2]$. Let

$$g_t(z) := f_{t-\alpha(t)}(e^{\alpha(t)}z), \quad z \in \mathbb{B}^n, \quad t \geq 0.$$

Then $(g_t)_{t \geq 0}$ is a normalized regular family with initial element f_0 . Moreover, notice that $g_t = f_t$ for $t \in [0, +\infty) \setminus (T_1, T_2)$. Let $T_1 < T'_1 < T'_2 < T_2$ be fixed. By a direct computation, we have for all $z \in \mathbb{B}^n$ and a.e. $t \geq 0$

$$(2.10) \quad [d(g_t)_z]^{-1} \frac{\partial g_t}{\partial t}(z) = e^{-\alpha(t)}(1 - \alpha'(t)) [df_{t-\alpha(t)}]^{-1} \frac{\partial f_{t-\alpha(t)}}{\partial t}(e^{\alpha(t)}z) + \alpha'(t)z$$

Notice that $A < 1$ implies $t - \alpha(t) \in [T_1, T_2]$ for $t \in [T_1, T_2]$. Therefore, setting $a(t) = 0$ for $t \in [0, +\infty) \setminus [T_1, T_2]$ and $a(t) = a$ for $t \in [T_1, T_2]$ and taking into account that $(f_t)_{t \geq 0}$ is exponentially squeezing in $[T_1, T_2]$ (with squeezing ratio $a \in (0, 1)$), we obtain from (2.10) that

$$\operatorname{Re} \left\langle [d(g_t)_z]^{-1} \frac{\partial g_t}{\partial t}(z), \frac{z}{\|z\|^2} \right\rangle \geq (1 - \alpha'(t))a(t) + \alpha'(t) \geq 0, \quad \forall z \in \mathbb{B}^n, \quad \text{a.e. } t \geq 0.$$

By Theorem 2.7.(3) and the fact that g_t is univalent for all $t \geq 0$, $(g_t)_{t \geq 0}$ is a normalized Loewner chain. Since $A < a$, it follows easily that $(g_t)_{t \geq 0}$ is exponentially squeezing in $[T'_1, T'_2]$.

Since $\alpha(t) \leq c < 0$ for all $t \in [T'_1, T'_2]$, it follows easily that $(g_t)_{t \geq 0}$ satisfies (1) of Definition 2.18 for all $t \in [T'_1, T'_2]$. From the definition of g_t , there exists a constant $c_1 > 0$ such that $\|d(g_t)_z\| \leq c_1$ for $z \in \mathbb{B}^n$, a.e. $t \in [T'_1, T'_2]$. Since \mathcal{M} is compact, using (2.10) and the fact that $[d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z) \in \mathcal{M}$, we conclude that there exists a constant $c_2 > 0$ such that $\|[d(g_t)_z]^{-1} \frac{\partial g_t}{\partial t}(z)\| \leq c_2$ for $z \in \mathbb{B}^n$, a.e. $t \in [T'_1, T'_2]$. Then, by Remark 2.20, $(g_t)_{t \geq 0}$ also satisfies (2) of Definition 2.18 and it is therefore geräumig in $[T'_1, T'_2]$.

Finally, if $(f_t)_{t \geq 0}$ is a normal Loewner chain, then (g_t) is a normal Loewner chain as well, because $g_t = f_t$ for $t \geq T_2$. \square

Remark 2.24. If (f_t) is a normalized/normal Loewner chain which is exponentially squeezing in $[T_1, \infty)$ for some $T_1 \geq 0$, then for every $m > T_1$ the previous result allows to construct a normalized/normal Loewner chain (g_t^m) which coincides with (f_t) on $\mathbb{R}^+ \setminus (T_1, m)$ and it is geräumig in (T'_1, T'_2) for all $T_1 < T'_1 < T'_2 < m$.

3. VARIATION OF GERÄUMIG LOEWNER CHAINS

Theorem 3.1. *Assume that $(f_t)_{t \geq 0}$ is a normalized Loewner chain, respectively a normal Loewner chain, on \mathbb{B}^n . If $(f_t)_{t \geq 0}$ is geräumig in $[0, T)$ for some $T > 0$, then there exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, setting*

$$\alpha(t) := \begin{cases} \epsilon \left(1 - \frac{t}{T}\right), & t \in [0, T) \\ 0, & t \in [T, +\infty) \end{cases}$$

the family $(f_t(z) + \alpha(t)h(z))_{t \geq 0}$ is a normalized Loewner chain, respectively a normal Loewner chain, on \mathbb{B}^n for every $h : \mathbb{B}^n \rightarrow \mathbb{C}^n$ holomorphic with $h(0) = dh_0 = 0$ and $\sup_{z \in \mathbb{B}^n} \|h(z)\| \leq 1$, $\sup_{z \in \mathbb{B}^n} \|dh_z\| \leq 1$.

Proof. Let $c \in (0, 1)$ be the squeezing ratio of $(f_t)_{t \geq 0}$ and let $a, b > 0$ be given by Definition 2.18. Up to replacing a and c with $\min\{a, c\}$, we can suppose $a = c$. Set $\epsilon_0 = \min \left\{ \frac{a}{2}, \frac{a^3 T}{2(a+bT)} \right\}$.

First of all, notice that $(f_t + \alpha(t)h)_{t \geq 0}$ is a normalized regular family.

Let $E \subset [0, +\infty)$ be a set of full measure for which all conditions in the hypotheses hold and such that $\frac{\partial f_t}{\partial t}$ exists for all $t \in E$. First, we notice that $d(f_t)_z + \alpha(t)dh_z$ is invertible for all $t \in E$ and all $z \in \mathbb{B}^n$. Indeed, it is so if $t \geq T$. If $t \in E \cap [0, T)$ then fix $z \in \mathbb{B}^n$ and let $v \in \mathbb{C}^n$, $\|v\| = 1$, be such that $\mu(d(f_t)_z + \alpha(t)dh_z) = \|[d(f_t)_z + \alpha(t)dh_z](v)\|$. It follows by (1) of Definition 2.18

$$\begin{aligned} \mu(d(f_t)_z + \alpha(t)dh_z) &= \|[d(f_t)_z + \alpha(t)dh_z](v)\| \geq \|d(f_t)_z(v)\| - \alpha(t)\|dh_z(v)\| \\ &\geq \mu(d(f_t)_z) - \alpha(t)\|dh_z\| \geq a - \alpha(t) > 0. \end{aligned}$$

Hence, we can well define a vector field $G(z, t)$, holomorphic in $z \in \mathbb{B}^n$ and measurable in $t \in [0, +\infty)$ in the following way

$$G(z, t) := \begin{cases} -[d(f_t)_z + \alpha(t)dh_z]^{-1} \left(\frac{\partial f_t}{\partial t}(z) + \alpha'(t)h(z) \right), & t \in E \\ 0 & t \in [0, +\infty) \setminus E. \end{cases}$$

If $t \in E$, then $G(z, t) = -z + \sum_{k \geq 2} Q_k(z, t)$ where Q_k is a polynomial mapping in z of order k . Hence $G(0, t) \equiv 0$ and $dG_0 = -\text{id}$. We want to show that $-G(\cdot, t) \in \mathcal{M}$ for all $t \in E$. For $t \geq T$ it is true because $(f_t)_{t \geq 0}$ is a normalized Loewner chain, so we have to check the condition for $t \in E \cap [0, T)$. To this aim, we first note that by Lemma 2.16, $\|\alpha(t)(d(f_t)_z)^{-1}dh_z\| \leq \alpha(t)/a \leq \alpha(0)/a \leq 1/2$ for all $t \in E \cap [0, T)$. Therefore, for $t \in E \cap [0, T)$

$$\begin{aligned} [d(f_t)_z + \alpha(t)dh_z]^{-1} &= [\text{id} + \alpha(t)(d(f_t)_z)^{-1}dh_z]^{-1}[d(f_t)_z]^{-1} \\ &= \sum_{j=0}^{\infty} (-1)^j \alpha(t)^j [(d(f_t)_z)^{-1}dh_z]^j [d(f_t)_z]^{-1}, \end{aligned}$$

and

$$\begin{aligned} \|[d(f_t)_z + \alpha(t)dh_z]^{-1}\| &\leq \sum_{j=0}^{\infty} \alpha(t)^j \|(d(f_t)_z)^{-1}\|^{j+1} \|dh_z\|^j \\ (3.1) \qquad \qquad \qquad &\leq \sum_{j=0}^{\infty} \frac{\alpha(t)^j}{a^{j+1}} \leq \frac{2}{a}. \end{aligned}$$

While,

$$\begin{aligned}
 (3.2) \quad \| [d(f_t)_z + \alpha(t)dh_z]^{-1} - [d(f_t)_z]^{-1} \| &\leq \sum_{j=1}^{\infty} \frac{\alpha(t)^j}{a^{j+1}} \\
 &= \frac{\alpha(t)}{a^2} \frac{1}{1 - \frac{\alpha(t)}{a}} \leq 2 \frac{\alpha(t)}{a^2} \leq 2 \frac{\epsilon_0}{a^2}.
 \end{aligned}$$

Now, since $\frac{\partial f_t(0)}{\partial t} = 0$ for all $t \in E$, by the Schwarz lemma and (2) of Definition 2.18 it holds $\left\| \frac{\partial f_t(z)}{\partial t} \right\| \leq b\|z\|$. Also, by the Schwarz lemma, $\|h(z)\| \leq \|z\|$. Hence, for all $t \in E \cap [0, T)$

$$\begin{aligned}
 &\operatorname{Re} \left\langle [d(f_t)_z + \alpha(t)dh_z]^{-1} \left(\frac{\partial f_t}{\partial t}(z) + \alpha'(t)h(z) \right), \frac{z}{\|z\|^2} \right\rangle \\
 &= \operatorname{Re} \left\langle [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z), \frac{z}{\|z\|^2} \right\rangle \\
 &+ \operatorname{Re} \left\langle ([d(f_t)_z + \alpha(t)dh_z]^{-1} - [d(f_t)_z]^{-1}) \frac{\partial f_t}{\partial t}(z), \frac{z}{\|z\|^2} \right\rangle \\
 &+ \operatorname{Re} \left\langle [d(f_t)_z + \alpha(t)dh_z]^{-1} \alpha'(t)h(z), \frac{z}{\|z\|^2} \right\rangle \\
 &\geq a - \| [d(f_t)_z + \alpha(t)dh_z]^{-1} - [d(f_t)_z]^{-1} \| \left\| \frac{\partial f_t}{\partial t}(z) \right\| \frac{1}{\|z\|} \\
 &- |\alpha'(t)| \| [d(f_t)_z + \alpha(t)dh_z]^{-1} \| \|h(z)\| \frac{1}{\|z\|} \geq a - \frac{2b\epsilon_0}{a^2} - \frac{2\epsilon_0}{aT} \geq 0,
 \end{aligned}$$

which proves that $\operatorname{Re} \left\langle G(z, t), \frac{z}{\|z\|^2} \right\rangle \leq 0$ for all $t \in E$ and $z \in \mathbb{B}^n \setminus \{0\}$ and by Remark 2.1, $-G(\cdot, t) \in \mathcal{M}$ for all $t \in E$.

Hence, $G(z, t)$ is a Herglotz vector field associated with the class \mathcal{M} in \mathbb{B}^n and $(f_t + \alpha(t)h)_{t \geq 0}$ is a normalized solution to the Loewner-Kufarev PDE associated to $G(z, t)$. In particular, it is a subordination chain by Theorem 2.7. In order to prove that it is a normalized Loewner chain, we only need to show that $f_t + \alpha(t)h$ is univalent for all $t \geq 0$.

Let $(g_t)_{t \geq 0}$ be the canonical solution associated to $G(z, t)$. By Theorem 2.7, there exists a holomorphic mapping $\Phi : \mathbb{C}^n \rightarrow \cup_{t \geq 0} (f_t + \alpha(t)h)(\mathbb{B}^n)$ such that $f_t(z) + \alpha(t)h(z) = \Phi(g_t(z))$ for all $t \geq 0$ and $z \in \mathbb{B}^n$. Note that, for $t \geq T$, $\alpha(t) \equiv 0$, hence $f_t(z) = \Phi(g_t(z))$ for all $t \geq T$. Taking into account that $\cup_{t \geq T} g_t(\mathbb{B}^n) = \mathbb{C}^n$ and f_t is univalent for all $t \geq 0$, it is easy to see that Φ is univalent. Therefore, $(f_t + \alpha(t)h)_{t \geq 0}$ is a normalized Loewner chain.

Finally, note that $\{e^{-t}(f_t + \alpha(t)h)\}_{t \geq 0}$ is a normal family if and only if $\{e^{-t}f_t\}_{t \geq 0}$ is. \square

Not all “nice” mappings in the class S^0 can be embedded into a normal Loewner chain which is geräumig in $[0, T)$ for some $T > 0$, as the following example shows:

Example 3.2. Let $a \in \mathbb{C}$ and let $f_a(z_1, z_2) := (z_1 + az_2^2, z_2)$. Note that f_a is an automorphism of \mathbb{C}^2 . Let $g_a := f_a|_{\mathbb{B}^2}$. It is known that $g_a \in S^0$ for $|a| \leq 3\sqrt{3}/2$ (see [36, Example 3]), while $g_a \notin S^0$ for $|a| > 2\sqrt{15}$ (see [14, Remark 3.5]). Let $r_0 := \sup\{r \geq 0 : g_r \in S^0\}$. Then $3\sqrt{3}/2 \leq r_0 \leq 2\sqrt{15}$. Since S^0 is compact, $g_{r_0} \in S^0$. If g_{r_0} were embeddable in a normal Loewner chain which is geräumig in $[0, T)$ for some $T > 0$, then by Theorem 3.1 there would exist $\epsilon > 0$ such that $g_{r_0+\epsilon} \in S^0$, contradicting the definition of r_0 . Note however that g_{r_0} is embeddable into a normalized Loewner chain (which is not a normal Loewner chain) which is geräumig in $[T_1, T_2)$ for any $0 \leq T_1 < T_2 < \infty$ given by $g_t(z) := f_{r_0}(e^t z)$.

In a recent paper [5], the first named author proved that, in fact, $r_0 = \frac{3\sqrt{3}}{2}$ and g_{r_0} is a support point of the class S^0 , so that, according to Proposition 1.1, g_{r_0} cannot be embedded into any normal Loewner chain which is exponentially squeezing in some interval of \mathbb{R}^+ .

4. SUPPORT POINTS AND EXTREME POINTS

Definition 4.1. (i) A mapping $f \in S^0$ is called a *support point* if there exists a linear operator $L : \text{Hol}(\mathbb{B}^n, \mathbb{C}^n) \rightarrow \mathbb{C}$ which is continuous with respect to the topology of uniform convergence on compacta of $\text{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ and not constant on S^0 such that $\max_{g \in S^0} \text{Re } L(g) = \text{Re } L(f)$. We denote by $\text{Supp}(S^0)$ the set of support points of S^0 .

(ii) A mapping $f \in S^0$ is called an *extreme point* if $f = tg + (1-t)h$, where $t \in (0, 1)$, $g, h \in S^0$, implies $f = g = h$. We denote by $\text{Ex}(S^0)$ the set of extreme points of S^0 .

Lemma 4.2. Let L be a bounded linear operator on $\text{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ which is not constant on S^0 . Then there exists a polynomial mapping $h : \mathbb{B}^n \rightarrow \mathbb{C}^n$, $h(0) = 0$, $dh_0 = 0$, such that $\sup_{z \in \mathbb{B}^n} \|h(z)\| \leq 1$, $\sup_{z \in \mathbb{B}^n} \|dh_z\| \leq 1$ and $\text{Re } L(h) > 0$.

Proof. Since L is not constant, there exists $f \in S^0$ such that $L(z) \neq L(f(z))$. If $f = z + \sum_{j \geq 2} P_j(z)$ is the power series expansion of f at 0, then $\sum_{j \geq 2} L(P_j(z)) \neq 0$, hence there exists P_j such that $L(P_j) \neq 0$. Up to multiplication by a suitable complex number λ , we obtain the result. \square

Although our variational method in Theorem 3.1 works only for normalized Loewner chains which are geräumig in an interval $[0, T)$, $T > 0$, it allows to prove the following result:

Lemma 4.3. Let $(f_t)_{t \geq 0}$ be a normal Loewner chain which is geräumig in $[T_1, T_2)$ for some $0 \leq T_1 < T_2 \leq +\infty$. Then $f_0 \notin \text{Supp}(S^0) \cup \text{Ex}(S^0)$.

Proof. By [15, Thm. 2.1] and [35, Thm. 1.1], if f_0 is an extreme point or a support point, then so is $e^{-T_1} f_{T_1}$. Thus, it is enough to prove that $e^{-T_1} f_{T_1} \notin \text{Supp}(S^0) \cup \text{Ex}(S^0)$. By

Remark 2.21, $e^{-T_1}f_{T_1}$ is embeddable into a normal Loewner chain which is geräumig in $[0, T_2 - T_1]$, therefore, we can assume with no loss of generality that $T_1 = 0$.

Let L be a bounded linear operator on $\text{Hol}(\mathbb{B}^n, \mathbb{C}^n)$ which is not constant on S^0 and let h be given by Lemma 4.2. By Theorem 3.1, there exists $\epsilon > 0$ such that $f_0 \pm \epsilon h \in S^0$. But $\text{Re } L(f_0 + \epsilon h) = \text{Re } L(f_0) + \epsilon \text{Re } L(h) > \text{Re } L(f_0)$, and f_0 is not a support point for L .

Also, since

$$f_0 = \frac{1}{2}(f_0 + \epsilon h) + \frac{1}{2}(f_0 - \epsilon h),$$

f_0 is not an extreme point of S^0 . □

Proposition 1.1 follows then at once from Theorem 2.23 and Lemma 4.3.

We give some applications of Proposition 1.1.

Corollary 4.4. *Let $g \in S^0$ and let $r \in (0, 1)$. Also, let $f(z) := \frac{1}{r}g(rz)$. Then $f \in S^0$ and $f \notin \text{Supp}(S^0) \cup \text{Ex}(S^0)$.*

Proof. Let $(g_t)_{t \geq 0}$ be a normal Loewner chain such that $g_0 = g$. Set $f_t(z) := \frac{1}{r}g_t(rz)$. Then $(f_t)_{t \geq 0}$ is a normal Loewner chain such that $f_0 = f$, which thus belongs to S^0 .

Next, recall that $\{g_t\}_{t \geq 0}$ satisfies the Loewner PDE

$$(4.1) \quad \frac{\partial g_t}{\partial t}(z) = -d(g_t)_z G(z, t), \quad \text{a.e. } t \geq 0, \forall z \in \mathbb{B}^n,$$

where $G(z, t)$ is a Herglotz vector field in \mathbb{B}^n associated with the class \mathcal{M} .

Finally, since $-G(z, t)$ is in the class \mathcal{M} , for a.e. $t \geq 0$, it follows from (4.1) that there exists $c > 0$ (depending only on r) such that for all $\|z\| \leq r$ and a.e. $t \geq 0$

$$(4.2) \quad \text{Re} \left\langle [d(g_t)_z]^{-1} \frac{\partial g_t}{\partial t}(z), \frac{z}{\|z\|^2} \right\rangle \geq c.$$

Indeed, let $E \subset [0, +\infty)$ be a set of full measure such that $G(z, t)$ is a Herglotz vector field associated with the class \mathcal{M} , namely, $\text{Re} \left\langle G(z, t), \frac{z}{\|z\|^2} \right\rangle < 0$ for all $z \in \mathbb{B}^n$ and $t \in E$. Suppose by contradiction (4.2) does not hold for some sequence $\{t_k\}_{k \in \mathbb{N}} \subset E$. Then by (4.1) there exists a sequence $\{z_k\}_{k \in \mathbb{N}} \subset \mathbb{B}(0, r)$, which we may suppose convergent to some z_0 , with $\|z_0\| \leq r$ such that $\text{Re} \left\langle G(z_k, t_k), \frac{z_k}{\|z_k\|^2} \right\rangle > -1/k$. Since $G(0, t_k) = 0$ and $dG(0, t_k) = -\text{id}$, clearly $z_0 \neq 0$. Let $v_k := z_k/\|z_k\|$ and consider the holomorphic functions $g_k : \mathbb{D} \rightarrow \mathbb{C}$ defined by

$$g_k(\zeta) := \langle G(\zeta v_k, t_k), v_k \rangle.$$

Then $g_k(\zeta) = -\zeta p_k(\zeta)$ where $\text{Re } p_k(\zeta) > 0$ for all $\zeta \in \mathbb{D}$ and $p_k(0) = 1$, for all $k \in \mathbb{N}$. Then $\{p_k\}_{k \in \mathbb{N}}$ is a sequence of functions in the Carathéodory class. In particular, since such a class is compact, we can assume that $p_k \rightarrow p$ for some holomorphic function $p : \mathbb{D} \rightarrow \mathbb{C}$ with $p(0) = 1$ and $\text{Re } p(\zeta) > 0$ for all $\zeta \in \mathbb{D}$. But $\text{Re } p_k(\|z_k\|) \rightarrow \text{Re } p(\|z_0\|)$, which forces $\text{Re } p(\|z_0\|) = 0$, a contradiction. Hence (4.2) holds.

Now, since $[d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z) = \frac{1}{r} [d(g_t)_{rz}]^{-1} \frac{\partial g_t}{\partial t}(rz)$, it follows from (4.2) that for all $z \in \mathbb{B}^n \setminus \{0\}$ and a.e. $t \geq 0$, it holds

$$\operatorname{Re} \left\langle [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z), \frac{z}{\|z\|^2} \right\rangle = \operatorname{Re} \left\langle [d(g_t)_{rz}]^{-1} \frac{\partial g_t}{\partial t}(rz), \frac{rz}{\|rz\|^2} \right\rangle \geq c.$$

Hence $(f_t)_{t \geq 0}$ is exponentially squeezing in $[0, +\infty)$. By Proposition 1.1, it follows that $f \notin \operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0)$. \square

Let $c \in [0, 1)$. We recall [38] that a normalized locally biholomorphic mapping $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$ is said to be *almost starlike of order c* if

$$\operatorname{Re} \langle [df_z]^{-1} f(z), z \rangle > c \|z\|^2, \quad z \in \mathbb{B}^n \setminus \{0\}.$$

Corollary 4.5. *Let $f \in S^0$ be an almost starlike mapping of order $c \in (0, 1)$. Then $f_0 \notin \operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0)$.*

Proof. Let

$$f(z, t) = e^t f(z), \quad z \in \mathbb{B}^n, \quad t \geq 0.$$

Since f is almost starlike of order c , $(f_t)_{t \geq 0}$ is a normal Loewner chain with initial element f , and we have

$$\begin{aligned} \operatorname{Re} \left\langle [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z), z \right\rangle &= \operatorname{Re} \langle [df_z]^{-1} f(z), z \rangle \\ &\geq c \|z\|^2, \quad z \in \mathbb{B}^n, \quad t \geq 0. \end{aligned}$$

Hence $(f_t)_{t \geq 0}$ is exponentially squeezing in $[0, +\infty)$. Thus, it is enough to apply Proposition 1.1. \square

A large class of mappings in S^0 which are not extreme/support points of S^0 may be obtained in the following way:

Corollary 4.6. *Let $(f_t)_{t \geq 0}$ be a normal Loewner chain and let $G(z, t)$ be the corresponding Herglotz vector field associated with the class \mathcal{M} . Assume that*

$$G(z, t) = -[\operatorname{id} - E(z, t)]^{-1} [\operatorname{id} + E(z, t)](z), \quad z \in \mathbb{B}^n, \quad t \geq 0,$$

where $E(z, t)$ is an $(n \times n)$ -matrix which is holomorphic with respect to $z \in \mathbb{B}^n$, $E(0, t) = 0$, for $t \geq 0$, and $E(z, t)$ is measurable with respect to $t \in [0, \infty)$, for $z \in \mathbb{B}^n$. If $\|E(z, t)\| \leq c < 1$ for $z \in \mathbb{B}^n$ and $t \geq 0$, then $f_0 \notin \operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0)$.

Proof. Since $\|E(z, t)\| \leq c$ and $E(0, t) = 0$, it follows that $\|E(z, t)\| \leq c\|z\|$, by the Schwarz lemma. Let

$$h(z, t) := [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z) = [\operatorname{id} - E(z, t)]^{-1} [\operatorname{id} + E(z, t)](z).$$

Then

$$h(z, t) - z = E(z, t)(h(z, t) + z), \quad z \in \mathbb{B}^n, \quad t \geq 0,$$

and thus

$$\|h(z, t) - z\|^2 \leq c^2 \|z\|^2 \|h(z, t) + z\|^2, \quad z \in \mathbb{B}^n, \quad t \geq 0.$$

Now, elementary computations yield that

$$\|z\|^2 \frac{1 - c\|z\|}{1 + c\|z\|} \leq \operatorname{Re} \langle h(z, t), z \rangle \leq \|z\|^2 \frac{1 + c\|z\|}{1 - c\|z\|}, \quad z \in \mathbb{B}^n, \quad t \geq 0,$$

and thus $(f_t)_{t \geq 0}$ is exponentially squeezing in $[0, +\infty)$. The result follows from Proposition 1.1. This completes the proof. \square

Corollary 4.7. *Let $f \in S$ be such that $\|df_z - \operatorname{id}\| \leq c$ for some $c \in (0, 1)$ and for all $z \in \mathbb{B}^n$. Then $f \in S^0$ and $f \notin \operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0)$.*

Proof. A normal Loewner chain with initial element f is given by $(f_t)_{t \geq 0}$ with $f_t(z) = f(e^{-t}z) + (e^t - e^{-t})z$ (see [12, Proof of Lemma 2.2]). A direct computation shows that

$$h(z, t) := [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z) = [\operatorname{id} - E(z, t)]^{-1} [\operatorname{id} + E(z, t)](z),$$

where $E(z, t) = e^{-2t}[\operatorname{id} - df_{e^{-t}z}]$. In view of the hypothesis, we deduce that $\|E(z, t)\| \leq c$, for all $z \in \mathbb{B}^n$ and $t \geq 0$, and thus the result follows from Corollary 4.6. \square

Remark 4.8. Both Corollary 4.5 and Corollary 4.7 can be proved directly without inspecting the natural normal Loewner chain. For instance, in Corollary 4.7, if L is a bounded linear functional not constant on S^0 and h is given by Lemma 4.2, it is easy to see that there exists $\epsilon > 0$ such that $\|df_z \pm \epsilon dh_z - \operatorname{id}\| < 1$ for all $z \in \mathbb{B}^n$. Hence by [12, Lemma 2.2], $f \pm \epsilon h \in S^0$ and then $f \notin \operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0)$.

Corollary 4.9. *Let $f : \mathbb{B}^n \rightarrow \mathbb{C}^n$ be a locally biholomorphic mapping such that $f(0) = 0$ and $df_0 = \operatorname{id}$. Assume that*

$$(4.3) \quad (1 - \|z\|^2) \| [df_z]^{-1} d^2(f)_z(z, \cdot) \| \leq c, \quad z \in \mathbb{B}^n,$$

for some $c \in (0, 1)$. Then $f \in S^0 \setminus (\operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0))$.

Proof. Pfaltzgraff [25, Theorem 2.4] proved that $f_t(z) = f(e^{-t}z) + (e^t - e^{-t})d(f)_{ze^{-t}}(z)$ is a normalized Loewner chain such that $\lim_{t \rightarrow \infty} e^{-t}f_t(z) = z$ locally uniformly on \mathbb{B}^n . Hence $(f_t)_{t \geq 0}$ is a normal Loewner chain, and thus $f \in S^0$. On the other hand, in view of the proof of [25, Theorem 2.4], we deduce that

$$h(z, t) := [d(f_t)_z]^{-1} \frac{\partial f_t}{\partial t}(z) = [\operatorname{id} - E(z, t)]^{-1} [\operatorname{id} + E(z, t)](z),$$

where

$$E(z, t) = -(1 - e^{-2t})[d(f)_{ze^{-t}}]^{-1} d^2(f)_{ze^{-t}}(ze^{-t}, \cdot), \quad z \in \mathbb{B}^n, \quad t \geq 0.$$

Taking into account the relation (4.3), it follows that $\|E(z, t)\| \leq c$, and thus the result follows from Corollary 4.6, as desired. \square

5. MAPPINGS THAT CAN BE EVOLVED IN FINITE TIME TO A BALL AND TIME- $\log M$ REACHABLE MAPPINGS

A natural class of mappings in S^0 where our construction applies is that of mappings whose image can be evolved to a ball:

Definition 5.1. Let f be a normalized univalent mapping in \mathbb{B}^n . We say that f can be evolved in finite time $N > 0$ to a ball if there exists a family $(f_t : \mathbb{B}^n \rightarrow \mathbb{C}^n)$ for $t \in [0, N]$ such that f_t is univalent for all $t \in [0, N]$, $f_s(\mathbb{B}^n) \subset f_t(\mathbb{B}^n)$ for all $0 \leq s \leq t \leq N$, $f_t(0) = 0$, $d(f_t)_0 = e^t \text{id}$, $f = f_0$ and $f_N(\mathbb{B}^n) = e^N \cdot \mathbb{B}^n$. We denote by \mathcal{E}_N the set of normalized univalent mappings in \mathbb{B}^n that can be evolved in finite time $N > 0$ to a ball.

Remark 5.2. Let f be a normalized univalent mapping in \mathbb{B}^n . Then it is easily seen that f can be evolved in finite time $N > 0$ to a ball if and only if there exists a normal Loewner chain $(f_t)_{t \geq 0}$ which is geräumig for $t > N$ such that $f = f_0$ and $f_N = e^N \text{id}$.

Proof. Indeed, if f can be evolved in finite time $N > 0$ to a ball by means of the family $(f_t)_{t \in [0, N]}$, then setting $f_t(z) = e^t z$ for $t \geq N$ and $z \in \mathbb{B}^n$, it is easy to see that the family $(f_t)_{t \geq 0}$ is a normal Loewner chain which is geräumig for $t > N$. The converse statement is obvious. \square

Moreover, let $M > 1$ and denote by

$$S^0(M) := \{f \in S^0 : \sup_{z \in \mathbb{B}^n} \|f(z)\| \leq M\}.$$

Then $\mathcal{E}_{\log M} \subset S^0(M)$, and also

$$\mathcal{E}_M \subset \mathcal{E}_{M'}, \quad \forall M < M'.$$

Remark 5.3. Let g be a normalized univalent function in the unit disc \mathbb{D} such that $\sup_{z \in \mathbb{D}} |g(z)| < R < +\infty$. Then $g \in \mathcal{E}_{\log R}$. The proof of this fact is a slight modification of Pommerenke's argument for the proof of embeddability of a given normalized univalent function in \mathbb{D} into a normalized Loewner chain (see [27, Thm. 6.1]). We sketch it here for the reader's convenience (see also [31], [32]).

Let $R > 1$ be such that $\sup_{z \in \mathbb{D}} |g(z)| < R$. Then $\overline{g(\mathbb{D})} \subset \mathbb{D}(0, R)$. Let $r \in (0, 1)$ and let $g_r(\zeta) := r^{-1}g(r\zeta)$. Then $g_r(\partial\mathbb{D})$ is a real analytic Jordan arc contained in $\mathbb{D}(0, R)$ for r close to 1. The domain $Q_r := \mathbb{D}(0, R) \setminus \overline{g_r(\mathbb{D})}$ is biholomorphic to an annulus $A = \{\zeta \in \mathbb{C} : \tau < |\zeta| < 1\}$ for some $0 < \tau < 1$. By Schwarz reflection principle, the biholomorphism $h : A \rightarrow Q_r$ extends holomorphically to the boundary and we can assume $h(\tau\partial\mathbb{D}) = g_r(\partial\mathbb{D})$, $h(\partial\mathbb{D}) = \partial\mathbb{D}(0, R)$. Now let $\Gamma_{r,t} := h((\tau + (1-\tau)t)\partial\mathbb{D})$ for $t \in [0, 1]$ and $\Gamma_{r,t} = tR\partial\mathbb{D}$ for $t > 1$. Let $H_{r,t}$ be the connected component of $\mathbb{C} \setminus \Gamma_{r,t}$ which contains 0. By construction, $\{H_{r,t}\}_{t \geq 0}$ is a family of simply connected domains containing 0 such that for all $0 \leq s < t$ the domain $H_{s,t}$ is properly contained in $H_{r,t}$ and $H_{r,t}$ kernel converges to $H_{r,s}$ for $t \rightarrow s$. Let $(g_{r,t})_{t \geq 0}$ be the normalized Loewner chain associated with $\{H_{r,t}\}$ (see [27, Ch. 6]). Then $g_{r,t}(0) = 0$, $g'_{r,t}(0) = e^t$, and there exists a strictly increasing absolutely

continuous function $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $g_{r,t}(\mathbb{D}) = H_{r,a(t)}$ for all $t \geq 0$. Moreover, since $H_{r,0} = g_r(\mathbb{D})$ and $H_{r,t} = \mathbb{D}(0, tR)$ for $t \geq 1$, it follows that $g_{r,0} = g_r$ and $g_{r,t}(\zeta) = e^t \zeta$ for all $t \geq \log R$. By [27, Lemma 6.2], there exists a subsequence $r_n \rightarrow 1$ such that for each fixed $t \geq 0$, $\{g_{r_n,t}\}$ converges uniformly on compacta of \mathbb{D} to a univalent function $g_t : \mathbb{D} \rightarrow \mathbb{C}$ and $(g_t)_{t \geq 0}$ is a normal Loewner chain. By construction, $g_0 = g$ and $f_t(\zeta) = e^t \zeta$ for all $t \geq \log R$ and for all $\zeta \in \mathbb{D}$.

The geometric notion of mappings that can be evolved in finite time to a ball has a counterpart in control theory. To see this, we first recall the following notion (see e.g. [30], [31], [32], [33], [15]):

Definition 5.4. Let f be a normalized univalent mapping in \mathbb{B}^n . We say that f is *time-log M -reachable* for some $M > 1$ if there exists a Herglotz vector field $G(z, t)$ associated with the class \mathcal{M} in \mathbb{B}^n such that $f = M\varphi(\cdot, \log M)$ where $(\varphi(z, t))_{t \geq 0}$ is the solution to the Loewner ODE (2.2) such that $\varphi(z, 0) = z$. The set of time-log M -reachable mappings generated by \mathcal{M} is denoted by $\tilde{\mathcal{R}}_{\log M}(\text{id}_{\mathbb{B}^n}, \mathcal{M})$.

By [15, Theorem 3.7], $f \in \tilde{\mathcal{R}}_{\log M}(\text{id}_{\mathbb{B}^n}, \mathcal{M})$ for some $M > 1$ if and only if f can be evolved in time $\log M$ to a ball, i.e.,

$$(5.1) \quad \tilde{\mathcal{R}}_{\log M}(\text{id}_{\mathbb{B}^n}, \mathcal{M}) = \mathcal{E}_{\log M}.$$

Thus, by [15, Corollary 3.8], for every $N > 0$, the set \mathcal{E}_N is compact.

Remark 5.5. According to [13, Corollary 7],

$$(5.2) \quad \tilde{\mathcal{R}}_{\log M}(\text{id}_{\mathbb{B}^n}, \mathcal{M}) \subset S^0(M) \setminus (\text{Supp}(S^0) \cup \text{Ex}(S^0)).$$

The geometrical counterpart of (5.2) follows at once either from (5.1) or directly from Remark 5.2 and Proposition 1.1:

Corollary 5.6. *Let f be a normalized univalent mapping in \mathbb{B}^n which can be evolved in finite time $\log M$ to a ball, for some $M > 1$. Then $f \in S^0(M) \setminus (\text{Supp}(S^0) \cup \text{Ex}(S^0))$.*

As we already remarked, in dimension one $S^0(M) = \mathcal{E}_{\log M} = \tilde{\mathcal{R}}_{\log M}(\text{id}_{\mathbb{D}}, \mathcal{M})$ for all $M > 1$. In higher dimension this is no longer the case. In order to properly state our result, we need a preliminary result (see [15, Example 3.5 and Theorem 3.12]).

Lemma 5.7. *Assume that $F \in S^0$ is starlike. For each $N > 1$, define*

$$(5.3) \quad F^N(z) = NF^{-1} \left(\frac{F(z)}{N} \right), \quad z \in \mathbb{B}^n.$$

Then $F^N \in \mathcal{E}_{\log N}$. Moreover, if F maximizes on S^0 a continuous functional $\lambda : S^0 \rightarrow \mathbb{R}$ then F^N maximizes on $\mathcal{E}_{\log N}$ the functional $\lambda^N : \mathcal{E}_{\log N} \rightarrow \mathbb{R}$, defined by

$$\lambda^N(g) = \lambda(NF(N^{-1}g(\cdot))), \quad g \in \mathcal{E}_{\log N}.$$

In addition, $\lambda^N(F^N) = \lambda(F)$.

Let

$$\Phi(z_1, z_2) := \left(z_1 + \frac{3\sqrt{3}}{2} z_2^2, z_2 \right).$$

Let $M := 2 \sup_{z \in \mathbb{B}^2} \|\Phi(z)\| < +\infty$. As we already remarked in Example 3.2, $\Phi \in S^0(M)$ is a support point for S^0 and, in fact, it maximizes the functional $\operatorname{Re} L_{0,2}^1 : S^0 \rightarrow \mathbb{R}$ given by $L_{0,2}^1(f) = \frac{1}{2} \frac{\partial^2 f_1}{\partial z_2^2}(0)$ for $f = (f_1, f_2) : \mathbb{B}^2 \rightarrow \mathbb{C}^2$ holomorphic mapping (see [5]). Such a mapping is starlike (see [36], [5]). Let $N > 1$. Then

$$(5.4) \quad \Phi^N(z) = \left(z_1 + \frac{3\sqrt{3}}{2} \left(1 - \frac{1}{N}\right) z_2^2, z_2 \right), \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Now we are ready to state and prove our result:

Theorem 5.8. *For any $N > 1$ the mapping Φ^N is a support point in $\mathcal{E}_{\log N}$ which maximizes the linear functional $\operatorname{Re} L_{0,2}^1$. In particular, for all $f = (f_1, f_2) \in \mathcal{E}_{\log N}$ it follows*

$$(5.5) \quad \left| \frac{\partial^2 f_1}{\partial z_2^2}(0) \right| \leq 3\sqrt{3} \left(1 - \frac{1}{N}\right),$$

and this estimate is sharp. Moreover, for any $2 < R < N$,

$$\Phi^N \in S^0(M) \cup \mathcal{E}_{\log N} \setminus (\mathcal{E}_{\log R} \cup \operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0)).$$

Proof. In view of Lemma 5.7, $\Phi^N \in \mathcal{E}_{\log N}$ maximizes the functional $(\operatorname{Re} L_{0,2}^1)^N : \mathcal{E}_{\log N} \rightarrow \mathbb{R}$. A direct computation shows that, given $f = (f_1, f_2) \in \mathcal{E}_{\log N}$,

$$(\operatorname{Re} L_{0,2}^1)^N(f) = \operatorname{Re} \frac{1}{2} \frac{\partial^2 f_1}{\partial z_2^2}(0) + \frac{3\sqrt{3}}{2N} = \operatorname{Re} L_{0,2}^1(f) + \frac{3\sqrt{3}}{2N}.$$

Hence, Φ^N maximizes $\operatorname{Re} L_{0,2}^1$ on $\mathcal{E}_{\log N}$ and (5.5) follows at once.

In order to prove the last statement, we notice that, given $1 < R < N$, setting $r = (1 - 1/N)$, it follows that $\Phi^N(z) = \frac{1}{r} \Phi(rz)$ for all $z \in \mathbb{B}^2$. Hence, if $r \in (1/2, 1)$ —which amounts to $R > 2$ —we have

$$\sup_{z \in \mathbb{B}^2} \|\Phi^N(z)\| \leq 2 \sup_{z \in \mathbb{B}^2} \|\Phi(z)\| = M.$$

Therefore, $\Phi^N \in S^0(M) \cup \mathcal{E}_{\log N}$. Also, by Corollary 4.4, $\Phi^N \notin (\operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0))$. Finally, $\Phi^N \notin \mathcal{E}_{\log R}$ for all $R < N$ because otherwise it would contradict (5.5). \square

Note that the mapping Φ is a bounded support point in S^0 which, by Corollary 5.6, cannot be evolved in finite time to a ball, and hence it is not time-log M -reachable. Therefore, such a mapping gives also a counterexample to Conjecture 3.9 in [15].

Moreover, and more interesting, Theorem 5.8 shows that, contrarily to the one-dimensional case, in general

$$\tilde{\mathcal{R}}_{\log M}(\operatorname{id}_{\mathbb{B}^n}, \mathcal{M}) \neq S^0(M) \setminus (\operatorname{Supp}(S^0) \cup \operatorname{Ex}(S^0)).$$

It is also interesting to note that for every $R > N$,

$$\Phi^N \in \mathcal{E}_{\log R} \setminus (\text{Supp}(\mathcal{E}_{\log R}) \cup \text{Ex}(\mathcal{E}_{\log R})).$$

Indeed, in view of Theorem 5.8, $\Phi^N \in \mathcal{E}_{\log N} \subset \mathcal{E}_{\log R}$. Moreover, given any bounded linear functional L on $\text{Hol}(\mathbb{B}^n, \mathbb{C}^n)$, it holds by linearity that $L(\Phi^N) = L(\text{id}) + c_L \cdot L_{0,2}^1(\Phi^N)$ for some c_L . Hence, again by Theorem 5.8, $\Phi^N \notin \text{Supp}(\mathcal{E}_{\log R})$. Moreover, let $g(z_1, z_2) = (\delta z_2^2, 0)$ for some $\delta > 0$. Note that $\Phi^N \pm g = \Phi^{N_\pm}$ for some $N_\pm > 1$. If $\delta < 1$ then $\Phi^N + g, \Phi^N - g \in \mathcal{E}_{\log R}$. Since $\Phi^N = \frac{1}{2}[(\Phi^N - g) + (\Phi^N + g)]$, it follows that $\Phi^N \notin \text{Ex}(\mathcal{E}_{\log R})$.

Remark 5.9. It is not known whether Φ , (respectively Φ^N), is an extreme point for S^0 (respectively for $\mathcal{E}_{\log N}$). However, since the hyperplane $\{g \in \text{Hol}(\mathbb{B}^2, \mathbb{C}^2) : \text{Re } L_{0,2}^1(g) = 3\sqrt{3}/2\}$ intersects S^0 and it is a separating hyperplane (see e.g. [21, Theorem 4.6]), there exists $f \in \text{Ex}(S^0)$ such that $\text{Re } L_{0,2}^1(f) = 3\sqrt{3}/2$. Similarly, for every $M > 1$ there exists $f \in \text{Ex}(\mathcal{E}_{\log M})$ such that $\text{Re } L_{0,2}^1(f) = \frac{3\sqrt{3}}{2}(1 - 1/M)$.

The previous considerations make natural to ask the following questions:

Question 5.10. *Let $M > 1$ and let $f \in S^0(M) \setminus (\text{Supp}(S^0) \cup \text{Ex}(S^0))$.*

- (1) *Is it true that there exists $R \geq M$ such that $f \in \mathcal{E}_{\log R}$?*
- (2) *Is it true that f can be embedded into an exponentially squeezing Loewner chain?*

Clearly, an affirmative answer to Question 5.10 (1) would imply an affirmative answer to Question 5.10 (2).

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